A global uniqueness result for acoustic tomography of moving fluid

Alexey Agaltsov

A global uniqueness result for acoustic tomography of moving fluid

A. D. Agaltsov[1,2]

We consider a model time-harmonic wave equation of acoustic tomography of moving fluid in an open bounded domain in dimension $d \geq 2$. We give global uniqueness theorems for related inverse boundary value problem at fixed frequency.

Keywords: inverse boundary value problems, time-harmonic wave equation, acoustic tomography

Subjects: partial differential equations, mathematical physics

AMS classification: 35R30 (Inverse problems), 35Q35 (PDEs in connection with fluid mechanics)

1 Introduction

Consider the operator

$$L_{A,V} = -\Delta - 2iA(x) \cdot \nabla + V(x), \tag{1}$$

where Δ is the standard Laplacian, $x \in D$, $A \in W^{1,\infty}(D, \mathbb{R}^d)$, $V \in L^\infty(D, \mathbb{R})$, D is an open bounded domain in $\mathbb{R}^d$ ($d \geq 2$). In the present article we study an inverse boundary value problem for the equation $L_{A,V}\psi = 0$ in D.

As in [AN], [RW], [RE] we consider the equation $L_{A,V}\psi = 0$ as a model equation for a time-harmonic ($e^{-i\omega t}$) pressure ψ in moving fluid. In this setting

$$A(x) = \frac{\omega}{c^2(x)} v(x), \quad V(x) = -\frac{\omega^2}{c^2(x)},$$

where v is the fluid velocity vector, c is the sound speed, ω is the frequency.

Suppose that 0 is not a Dirichlet eigenvalue for operator $L_{A,V}$ in D. Then the Dirichlet problem

$$\begin{cases} L_{A,V}\psi = 0 \quad \text{in } D, \\ \psi|_{\partial D} = f, \end{cases} \tag{2}$$

[1] Centre de Mathématiques Appliquées, Ecole Polytechnique
Route de Saclay
91128 PALAISEAU Cedex, France
[2] Lomonosov Moscow State University,
GSP-1, Leninskie Gory
119991 MOSCOW, Russia
email: agalets@gmail.com

is uniquely solvable for $\psi \in H^1(D)$ for any $f \in H^{1/2}(\partial D)$. The Dirichlet-to-Neumann map $\Lambda_{A,V}$ sends $f \in H^{1/2}(\partial D)$ to $\Lambda_{A,V} f \in H^{-1/2}(\partial D)$ defined by the formula

$$\Lambda_{A,V} f = \tfrac{\partial \psi}{\partial \nu}\big|_{\partial D} + i(A \cdot \nu) f, \tag{3}$$

where ν is the unit exterior normal to ∂D and $\frac{\partial \psi}{\partial \nu}\big|_{\partial D} \in H^{-1/2}(\partial D)$ can be defined, in particular, by the following formula:

$$\langle \tfrac{\partial \psi}{\partial \nu}\big|_{\partial D}, u \rangle = \int_D \Big(\nabla \psi(x) \nabla \tilde{u}(x) - 2i\tilde{u}(x) A(x) \nabla \psi(x) + \tilde{u}(x) V(x) \psi(x) \Big)\, dx, \tag{4}$$

for $u \in H^{1/2}(\partial D)$ and arbitrary $\tilde{u} \in H^1(D)$ with $\tilde{u}|_{\partial D} = u$. Note that since ψ satisfies $L_{A,V}\psi = 0$, the right hand side of the above formula doesn't depend on the choice of $\tilde{u}$.

The inverse boundary value problem for equation $L_{A,V}\psi = 0$ in D consists in finding A, V from $\Lambda_{A,V}$. In the case when coefficients A, V can be complex-valued there is an obstruction to the unique solvability of this problem caused by the gauge invariance of the map $\Lambda_{A,V}$ with respect to the gauge transformations

$$\begin{cases} A \to A + \nabla \varphi, \\ V \to V - i\Delta\varphi + (\nabla\varphi)^2 + 2A\nabla\varphi, \end{cases}$$

where $\varphi \in W^{2,\infty}(D, \mathbb{C})$, $\varphi|_{\partial D} = 0$, see, e.g., [KU] ($d \geq 3$), [GT] ($d = 2$).

However, in the case of real-valued coefficients A, V there is no gauge non-uniqueness as it was observed, for example, in [AN].

In addition, in general case, under some regularity assumptions on ∂D, A and V, the Dirichlet-to-Neumann map $\Lambda_{A,V}$ uniquely determines the two-form dA and the function q in D and the tangential component of A on ∂D, where

$$\begin{aligned} dA &= \sum_{1 \leq k < l \leq d} \left(\frac{\partial A^l}{\partial x_k} - \frac{\partial A^k}{\partial x_l} \right) dx_k \wedge dx_l, \\ q &= V + i\nabla \cdot A - A \cdot A, \end{aligned} \tag{5}$$

and $A = (A^1, \ldots, A^d)$.

In particular, it was shown in [KU] that in dimension $d \geq 3$ the map $\Lambda_{A,V}$ uniquely determines dA and q in D if $A \in L^\infty(D, \mathbb{C}^d)$ and $V \in L^\infty(D, \mathbb{C})$. And in dimension $d = 2$ it was shown in [GT] that if D is a smooth Riemann surface with boundary (in particular, if D is a planar domain with $\partial D \in C^\infty$) then the map $\Lambda_{A,V}$ uniquely determines dA and q provided that $A \in W^{2,p}(D, \mathbb{R}^d)$, $V \in W^{1,p}(D, \mathbb{C})$, $p > 2$.

In addition, concerning the identifiability of tangential components of A on the boundary, it was proved in [BS] that if $\partial D \in C^1$ ($d \geq 3$) or $\partial D \in C^{1,\alpha}$, $\alpha \in (0,1)$ ($d = 2$) and if $A \in C(\overline{D}, \mathbb{C}^d)$, $V \in L^\infty(D, \mathbb{C})$ then $\Lambda_{A,V}$ uniquely determines $A - \nu(A \cdot \nu)$ on ∂D, where ν is the unit exterior normal field to ∂D.

In the present article we combine the aforementioned results in order to obtain the following global uniqueness results in the case when coefficients A, V are real-valued.

Theorem 1. *Let D be a bounded simply connected domain with path connected boundary in $\mathbb{R}^d$ $(d \geq 3)$ with $\partial D \in C^1$. Let A_1, $A_2 \in W^{1,\infty}(D, \mathbb{R}^d)$ and V_1, $V_2 \in L^\infty(D, \mathbb{R})$. If $\Lambda_{A_1,V_1} = \Lambda_{A_2,V_2}$, then $A_1 = A_2$, $V_1 = V_2$.*

Theorem 2. *Let D be a bounded simply connected domain in $\mathbb{R}^2$ with $\partial D \in C^\infty$. Let A_1, $A_2 \in W^{2,p}(D, \mathbb{R}^d)$ and V_1, $V_2 \in W^{1,p}(D, \mathbb{R})$ with $p > 2$. If $\Lambda_{A_1,V_1} = \Lambda_{A_2,V_2}$ then $A_1 = A_2$ and $V_1 = V_2$.*

Theorems 1 and 2 are proved in Section 3. In Section 2 we present formulas and equations for finding A, V from dA, q, $A - \nu(A \cdot \nu)|_{\partial D}$.

2 Formulas and equations for finding A, V

In this section we suppose that D is a bounded contractible domain with path connected C^2 boundary in $\mathbb{R}^d$ $(d \geq 2)$. By contractibility we mean that there exists $F \in C^2(D \times [0,1], D)$ such that $F_0 \equiv \widetilde{x}$, $F_1 = \mathrm{id}_D$, where $F_t(x) = F(x,t)$, $\widetilde{x}$ is some fixed point in D and id_D is the identity mapping on D. We also suppose that $A \in W^{2,\infty}(D, \mathbb{R}^d)$, $V \in L^\infty(D, \mathbb{R})$. Given dA, q as in (5) in D and $A - \nu(A \cdot \nu)$ on ∂D, we can find A, V in the following way:

1. Define $\widetilde{A} = (\widetilde{A}^1, \ldots, \widetilde{A}^d) \in W^{1,\infty}(D, \mathbb{R}^d)$ by the formula

$$\widetilde{A}^k = \sum_{i<j} \int_0^1 \left(\frac{\partial F_t^i}{\partial t} \frac{\partial F_t^j}{\partial x_k} - \frac{\partial F_t^j}{\partial t} \frac{\partial F_t^i}{\partial x_k} \right) \left(\frac{\partial A^j}{\partial x_i} \circ F_t - \frac{\partial A^i}{\partial x_j} \circ F_t \right) dt, \tag{6}$$

   where $k = 1, \ldots, d$; $A = (A^1, \ldots, A^d)$, $F_t = (F_t^1, \ldots, F_t^d)$ and $\circ$ denotes the composition of maps, i.e. $\frac{\partial A^j}{\partial x_i} \circ F_t(y) = \frac{\partial A^j}{\partial x_i}(F_t(y))$, $y \in D$.

2. Fix $x^0 \in \partial D$. Define $\varphi_0 \in C^1(\partial D)$ by the formula

$$\varphi_0(x) = \sum_{k=1}^{d} \int_{x^0}^{x} \big(A_\tau^k(y) - \widetilde{A}_\tau^k(y)\big) dy_k, \quad x \in \partial D, \tag{7}$$

   where $A_\tau = A - \nu(A \cdot \nu)$, $\widetilde{A}_\tau = \widetilde{A} - \nu(\widetilde{A} \cdot \nu)$, $A_\tau = (A_\tau^1, \ldots, A_\tau^d)$, $\widetilde{A}_\tau = (\widetilde{A}_\tau^1, \ldots, \widetilde{A}_\tau^d)$, ν is the unit exterior normal field to ∂D and integration is over an arbitrary C^1 curve on ∂D linking x^0 to x.

3. Find the unique generalized solution $\varphi \in W^{2,\infty}(D, \mathbb{R})$ to

$$\begin{cases} \Delta\varphi = \operatorname{Im} q - \nabla \cdot \widetilde{A} & \text{in } D, \\ \varphi|_{\partial D} = \varphi_0. \end{cases} \tag{8}$$

4. Coefficients A, V are given by the following formulas:

$$\begin{cases} A = \widetilde{A} + \nabla\varphi, \\ V = q - i\Delta\varphi - i\nabla \cdot \widetilde{A} + \widetilde{A} \cdot \widetilde{A} + 2\widetilde{A} \cdot \nabla\varphi + (\nabla\varphi)^2. \end{cases}$$

This algorithm will be justified in Section 4.

3 Proofs of Theorems 1, 2

We will prove Theorems 1 and 2 simultaneously. Let D, A_1, A_2, V_1, V_2 satisfy the conditions of Theorem 1 (resp. Theorem 2) and suppose that $\Lambda_{A_1,V_1} = \Lambda_{A_2,V_2}$.

Using Theorem 1.1 of [BS] we obtain that

$$\big(A_1 - \nu(A_1\cdot\nu)\big)|_{\partial D} = \big(A_2 - \nu(A_2\cdot\nu)\big)|_{\partial D}, \tag{9}$$

where ν is the unit exterior normal field to ∂D. Using Theorem 1.1 of [KU] (resp. Theorem 1.1 of [GT]) we get

$$dA_1 = dA_2 \ \text{ in } D, \tag{10}$$

$$q_1 = q_2 \ \text{ in } D, \tag{11}$$

where

$$dA_j = \sum_{1\le k<l\le d}\left(\frac{\partial A_j^l}{\partial x_k} - \frac{\partial A_j^k}{\partial x_l}\right) dx_k \wedge dx_l,$$

$$q_j = V_j + i\nabla\cdot A_j - A_j\cdot A_j,$$

where $A_j = (A_j^1,\ldots,A_j^d)$, $j = 1, 2$.

Since the domain D is simply connected it follows from (10) that there exists $\varphi \in W^{2,\infty}(D,\mathbb{R})$ such that

$$A_1 - A_2 = \nabla\varphi \ \text{ in } D. \tag{12}$$

In dimension $d = 2$ it follows from simple connectedness of D and from smoothness of ∂D that ∂D is path connected. Formulas (9), (12) and path connectedness of ∂D imply that φ is constant on ∂D.

Using (11), (12) we obtain that

$$V_1 - V_2 = -i\Delta\varphi - (\nabla\varphi)^2 + 2A_1\nabla\varphi \ \text{ in } D.$$

Taking the imaginary part of this equation we obtain the equation $\Delta\varphi = 0$ in D. Since φ is constant on ∂D and $\varphi \in W^{2,\infty}(D)$ it follows that φ is constant in $\overline{D}$. Hence $A_1 = A_2$ and $V_1 = V_2$. Theorems 1 and 2 are proved.

4 Justification of the algorithm of Section 2

It follows from formula (6) that $d\widetilde{A} = dA$. More precisely, if we denote by F^*dA the pullback of the form dA by the map F and by ι_{∂_t} we denote the interior product with the vector field $\frac{\partial}{\partial t}$ on $\{(x,t)\in D\times[0,1]\}$, then

$$\sum_{k=1}^{d} \widetilde{A}^k dx_k = \int_0^1 (\iota_{\partial_t} F^* dA)\, dt,$$

and the equality $d\widetilde{A} = dA$ follows from the Cartan magic formula $\mathcal{L}_{\partial_t} = d \circ \iota_{\partial_t} + \iota_{\partial_t} \circ d$, where $\mathcal{L}_{\partial_t}$ is the Lie derivative along $\frac{\partial}{\partial t}$, d is the exterior derivative on $\{(x,t) \in D \times [0,1]\}$ and $\circ$ denotes the composition of maps.

Hence we can define $\varphi \in W^{2,\infty}(D,\mathbb{R})$ by the formula

$$\varphi(x) = \int_{\tilde{x}}^{x} \sum_{k=1}^{d} (A_k - \widetilde{A}_k) dx_k, \quad x \in \overline{D},$$

where $\tilde{x} \in D$ is some fixed point and integration is over an arbitrary C^1 curve in D linking $\tilde{x}$ to x. Then $\nabla\varphi = A - \widetilde{A}$ in $\overline{D}$ and this implies that $\varphi|_{\partial D}$ differs by constant from φ_0 defined in (7). We also obtain from (5) the equation

$$V = q - i\Delta\varphi - i\nabla \cdot \widetilde{A} + \widetilde{A} \cdot \widetilde{A} + 2\widetilde{A} \cdot \nabla\varphi + (\nabla\varphi)^2 \quad \text{in } D.$$

Taking into account that V is real-valued and separating the imaginary part in the latter equation we obtain (8). Since $\operatorname{Im} q$ and $\nabla \cdot \widetilde{A}$ belong to $L^\infty(D,\mathbb{R})$, the problem (8) is uniquely solvable for $\varphi \in W^{2,\infty}(D,\mathbb{R})$. Thus, the algorithm of Section 2 is justified.

5 Aknowledgements

This work was fulfilled within the framework of research carried out under the supervision of Prof. R. G. Novikov.

6 References

[AN] A. D. Agaltsov, R. G. Novikov, Riemann-Hilbert problem approach for two-dimensional flow inverse scattering, J. Math. Phys. **55** (10), 2014, 103502

[BS] R. M. Brown, M. Salo, Identifiability at the boundary for first-order terms, Appl. Anal. **85** (6–7), 2006, 735–749

[GT] C. Guillarmou, L. Tzou, Identification of a connection from Cauchy data on a Riemann surface with boundary, Geom. Funct. Anal. **21** (2), 2011, 393–418

[KU] K. Krupchyk, G. Uhlmann, Uniqueness in an inverse boundary problem for a magnetic Schrödinger operator with a bounded magnetic potential, Comm. Math. Phys. **327** (3), 2014, 993–1009

[RW] D. Roussef, K. B. Winters, Two-dimensional vector flow inversion by diffraction tomography, Inverse Problems **10**, 1994, 687–697

[RE] M. N. Rychagov, H. Ermert, Reconstruction of fluid motion in acoustic diffraction tomography, J. Acoust. Soc. Am. **99** (5), 1996, 3029–3035